Introduction

Basic Transformations

1. Translation
2. Rotation
3. Scaling
4. Reflection
5. shearing

HOMOGENOUS FORM

Expressing position in homogeneous coordinates allows us to represent all geometric transformations equation as matrix multiplications.

(x, y) -------------------- (x_h, y_h ,h)

$x= x_h /h$ $y= y_h /h$

where h is any non zero value

For convenient h=1

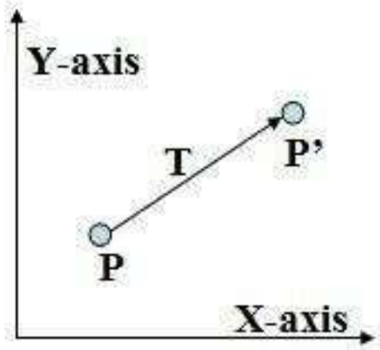
(x,y)------------------------(x, y, 1).

1. Translation

A translation is applied to an object by repositioning it along a straight-line path from one coordinate location to another. We translate a two-dimensional point by adding translation distances, tx, and ty, to the original coordinate position (x, y) to move the point to a new position (x ' , y').

x’ = x + tx

y’ = y + ty where the pair (tx, ty) is called the *translation vector* or *shift vector*.



*Translating a point from position **P** to position **P'** With translation vector **T**.*

1. Translation

We can write equation as a single matrix equation by using column vectors to represent coordinate points and translation vectors. Thus,

$$P = \begin{bmatrix} x \\ y \end{bmatrix} \qquad T = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \qquad P' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$P' = P + T$$

So we can write

In homogeneous representation if position P = (x, y) is translated to
new position P'= (x', y') then:

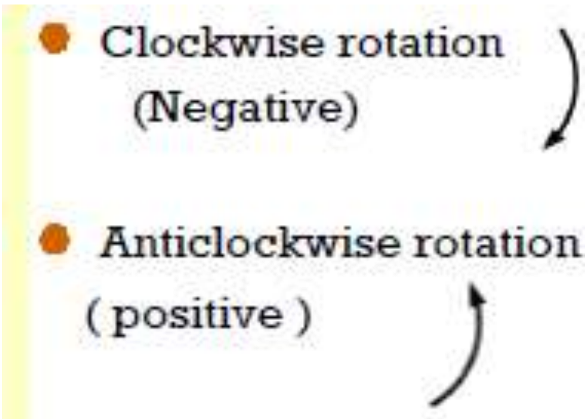
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = T(t_x, t_y).P$$

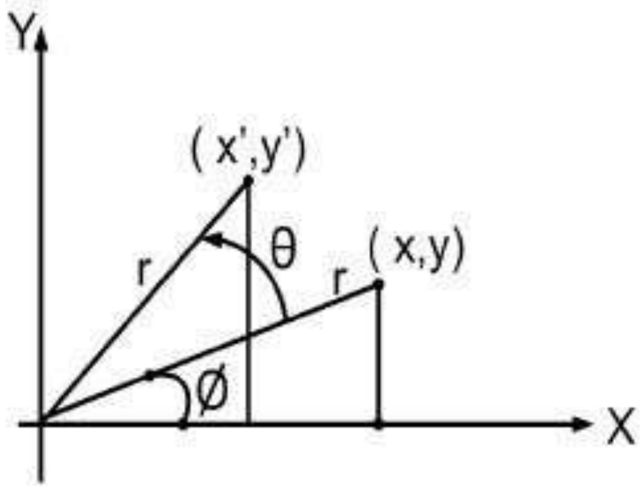
2. ROTATION

- A two-dimensional rotation is applied to an object by repositioning it along a circular path in the **xy** plane.
- To generate a rotation, we specify a rotation angle θ

  \+ Value for ‘θ’ define *counter-clockwise* rotation about a point

  \- Value for ‘θ’ defines *clockwise* rotation about a point



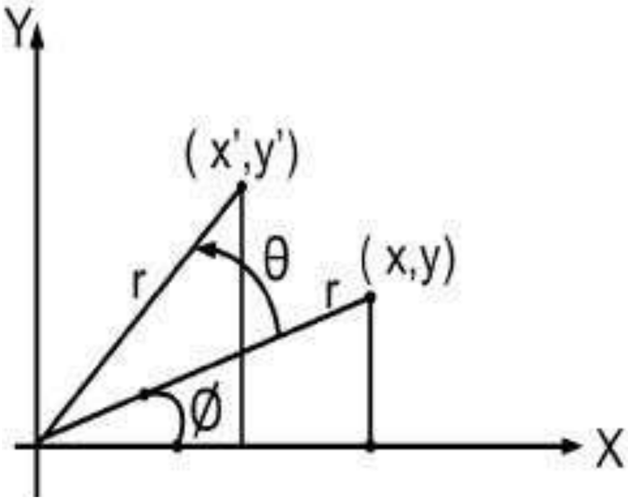


Anticlockwise direction

2. ROTATION

At Origin

Coordinates of point (x,y) in polar form



$$x = r\cos\phi, \qquad y = r\sin\phi$$

$$x' = r\cos(\phi+\theta) = r\cos\phi.\cos\theta - r\sin\phi.\sin\theta$$
$$y' = r\sin(\phi+\theta) = r\cos\phi.\sin\theta + r\sin\phi.\cos\theta$$

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$P' = R\,.\,P$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

ROTATION MATRX

2. ROTATION

In homogeneous co-ordinate

Anticlockwise direction

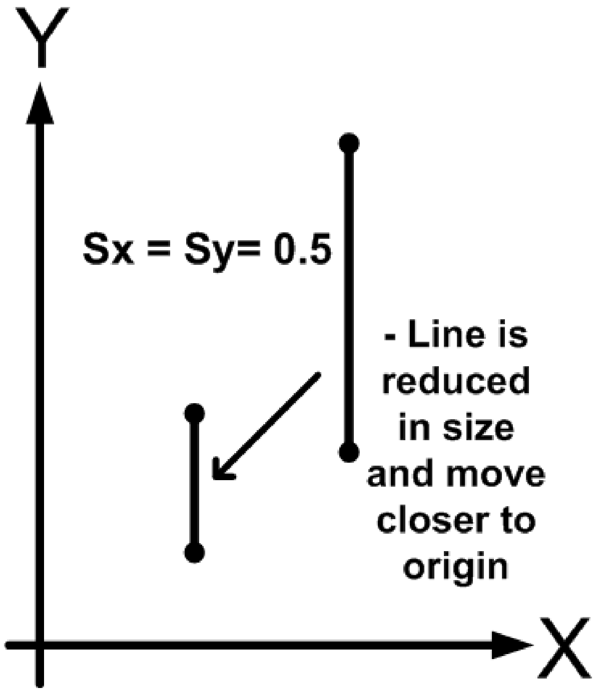
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

Clockwise direction

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = R(\theta).P$$

3.Scaling

- A scaling transformation alters the size of an object. This operation can be carried out for polygons by multiplying the coordinate values (x, y) of each vertex by scaling factors $\mathbf{S_x}$ and $\mathbf{S_y}$ to produce the transformed coordinates (x', y').
- s_x scales object in ‘x’ direction
- s_y scales object in ‘y’ direction



3.Scaling

Thus, for equation form,

$\mathbf{x' = x.\, s_x}$ and

$\mathbf{y' = y.\, s_y}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Values greater than 1 for s_x , s_y produce ***enlargement***
- Values less than 1 for s_x , s_y *reduce* size of object
- s x = s y = 1 leaves the size of the object ***unchanged***
- When s_x , s_y are assigned the same value s x = s y = 3 or 4 etc then a ***Uniform Scaling*** is produced

3.Scaling

$P' = S.P$

In homogeneous co-ordinate

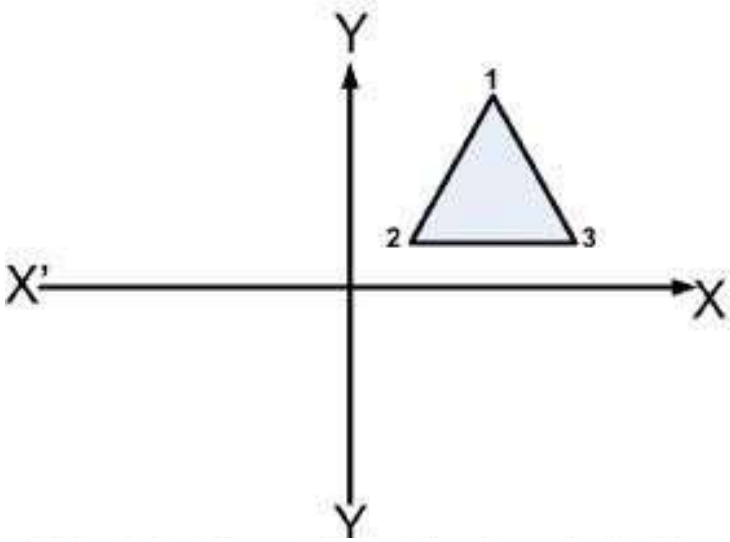
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow P' = S(s_x, s_y).P$$

4. Reflection

(i) Reflection about x axis or about line y = 0

Keeps **X** value same but flips **Y** value of coordinate points

x' = x
y' = -y



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

4. Reflection

(ii) Reflection about y axis or about line x = 0

Keeps ‘y’ value same but flips x value of coordinate points

$x' = -x$

$y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

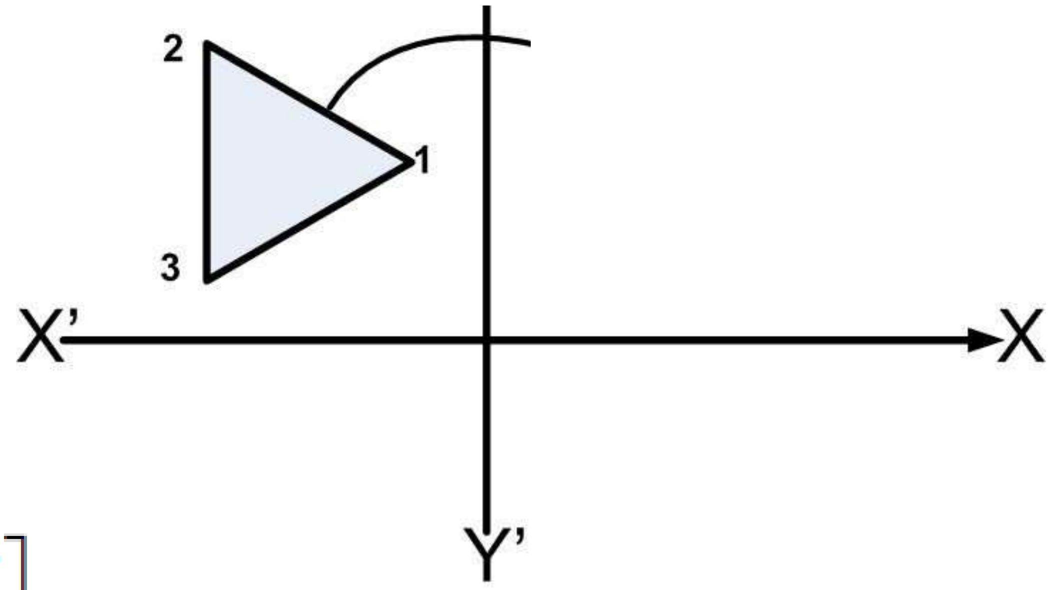


Fig: Reflection of object about y-axis (x=0)

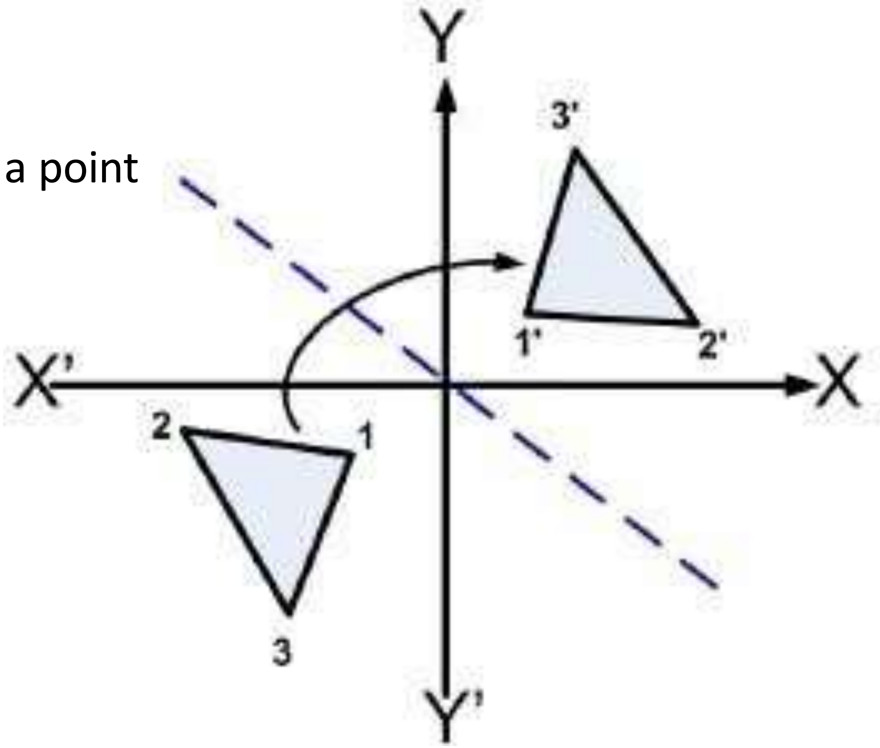
4. Reflection

(iii) Reflection about origin

Flip both ‘x’ and ‘y’ coordinates of a point

$x' = -x$
$y' = -y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



Homogeneous co-ordinate

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Reflection

(iv) Reflection about line y = x

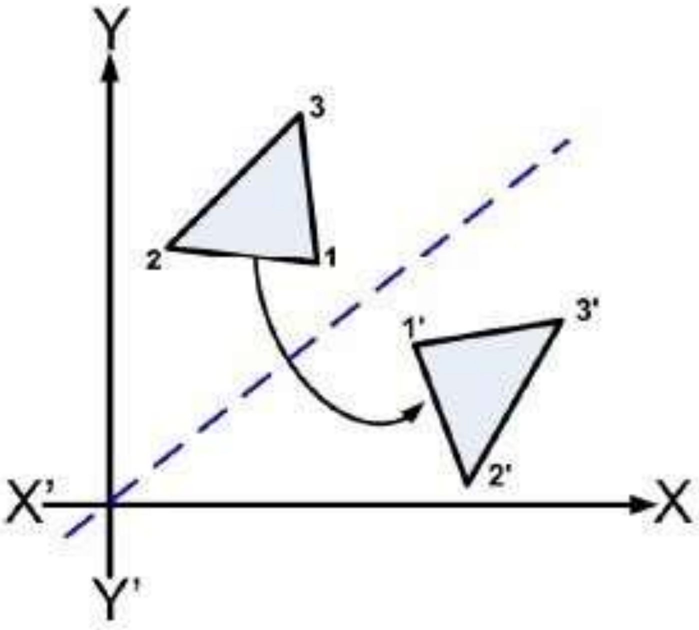
$x' = y$
$y' = x$

Thus, reflection against x=y-axis (i.e. θ =45)

$$R_{x=y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hence

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Homogeneous co-ordinate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

5. Shearing

- It distorts the shape of object in either ‘x’ or ‘y’ or both direction. In case of single directional shearing (e.g. in ‘x’ direction can be viewed as an object made up of very thin layer and slid over each other with the *base* remaining where it is). Shearing is a ***non-rigid-body transformation*** that moves objects with deformation.

5. Shearing

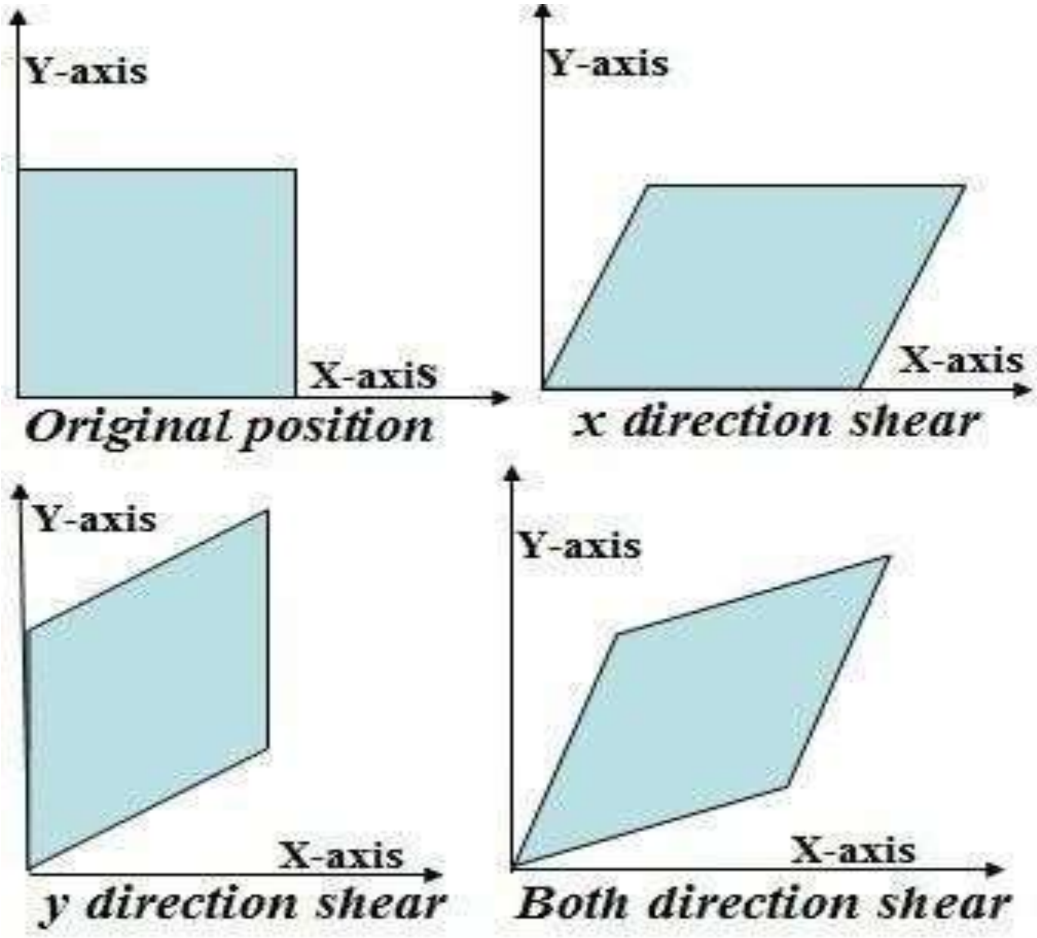


Fig. Two Dimensional shearing

5. Shearing

In 'x' direction,

$x' = x + S_{hx} \cdot y$

$y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & S_{hx} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In 'y' direction,

$x' = x$

$y' = y + S_{hy} \cdot x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ S_{hy} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In both directions,

$x' = x + S_{hx} \cdot y$

$y' = y + S_{hy} \cdot x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & S_{hx} \\ S_{hy} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Homogenous Coordinates

The matrix representations for translation, scaling and rotation are respectively:

- Translation: **P' = T + P** (*Addition*)
- Scaling: **P' = S . P** (*Multiplication*)
- Rotation: **P' = R . P** (*Addition*)

Since, the composite transformation such as include many sequence of translation, rotation etc and hence the many naturally differ addition & multiplication sequence have to perform by the graphics allocation. Hence, the applications will take more time for rendering.

Thus, we need to treat all three transformations in a consistent way so they can be combined easily & compute with one mathematical operation. If points are expressed in homogenous coordinates, all geometrical transformation equations can be represented as matrix multiplications.

Homogenous Coordinates

- **For translation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

With T(tx , ty) as translation matrix-

- **For rotation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} \text{Cos}\theta & -\text{Sin }\theta & 0 \\ \text{Sin}\theta & \text{Cos}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} \text{Cos}\theta & \text{Sin }\theta & 0 \\ -\text{Sin}\theta & \text{Cos}\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

(a) **(b)**

Here, figure-a shows the Counter Clockwise (CCW) rotation & figure-b shows the Clockwise (CW) rotation.

- **For scaling**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} S_{hx} & 0 & 0 \\ 0 & S_{hy} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- **For Reflection**
  - Reflection about x-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  - Reflection about y-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

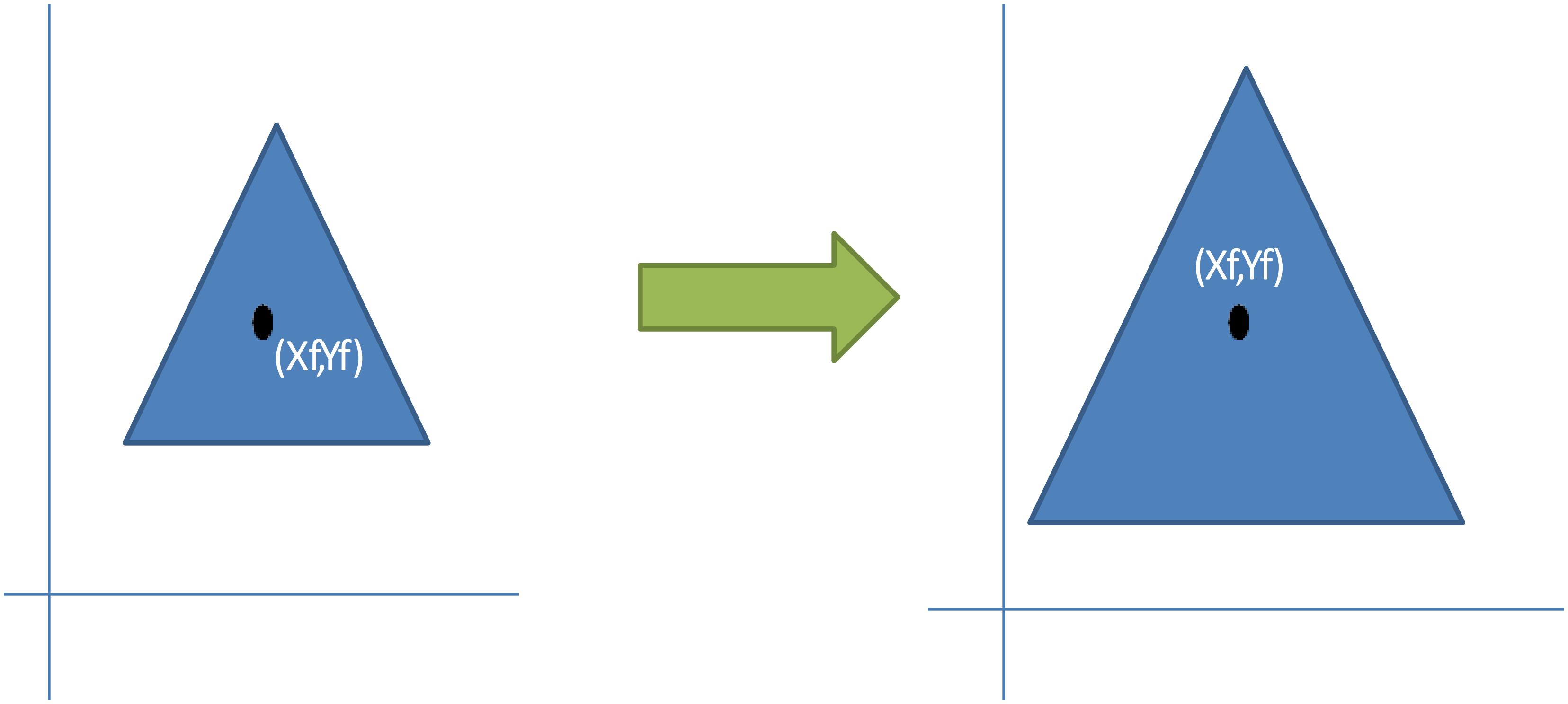
  - Reflection about y=x-axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

  - Reflection about y=-x-axis

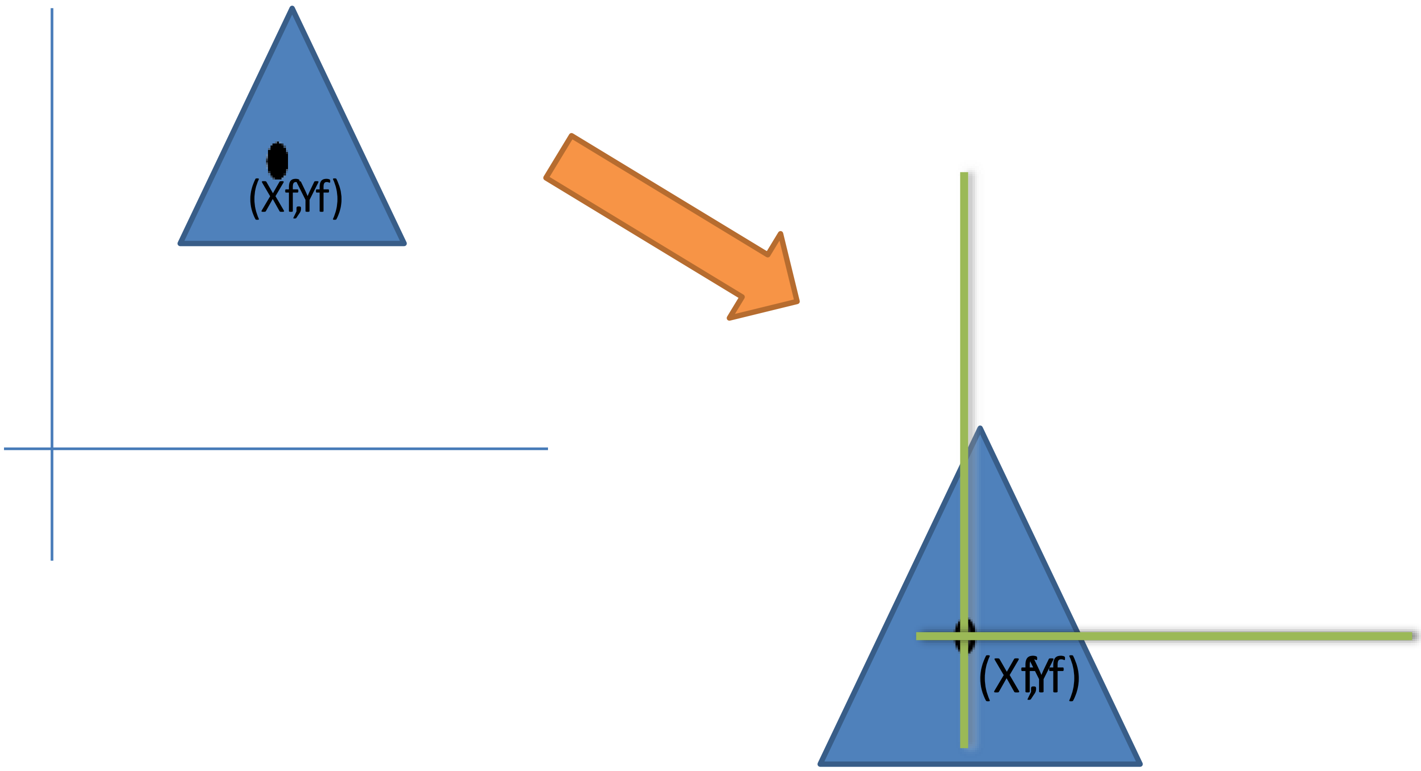
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Fixed Point Scaling



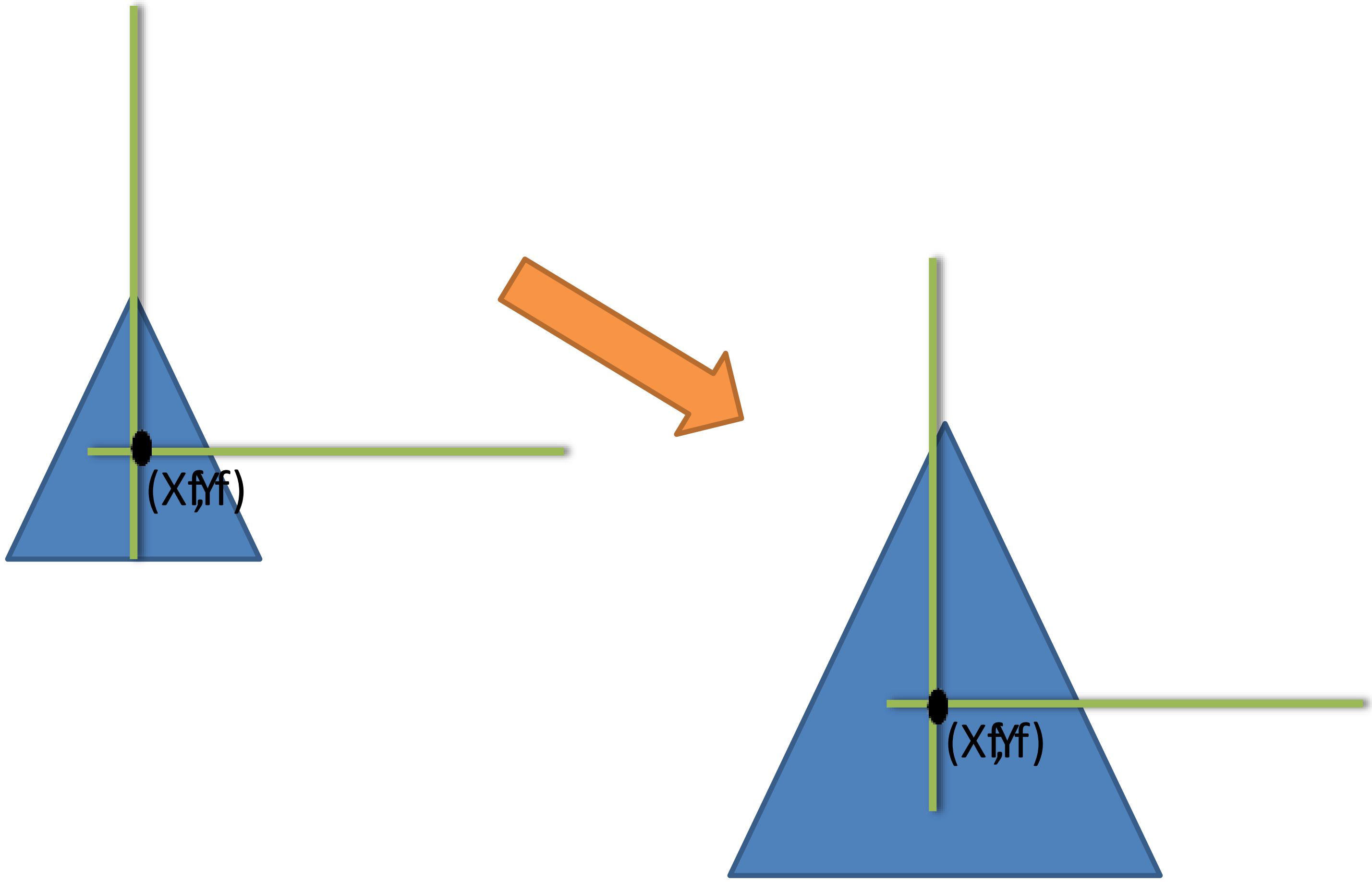
Fixed Point Scaling

Step 1: The fixed point along with the object is translated to coordinate origin.



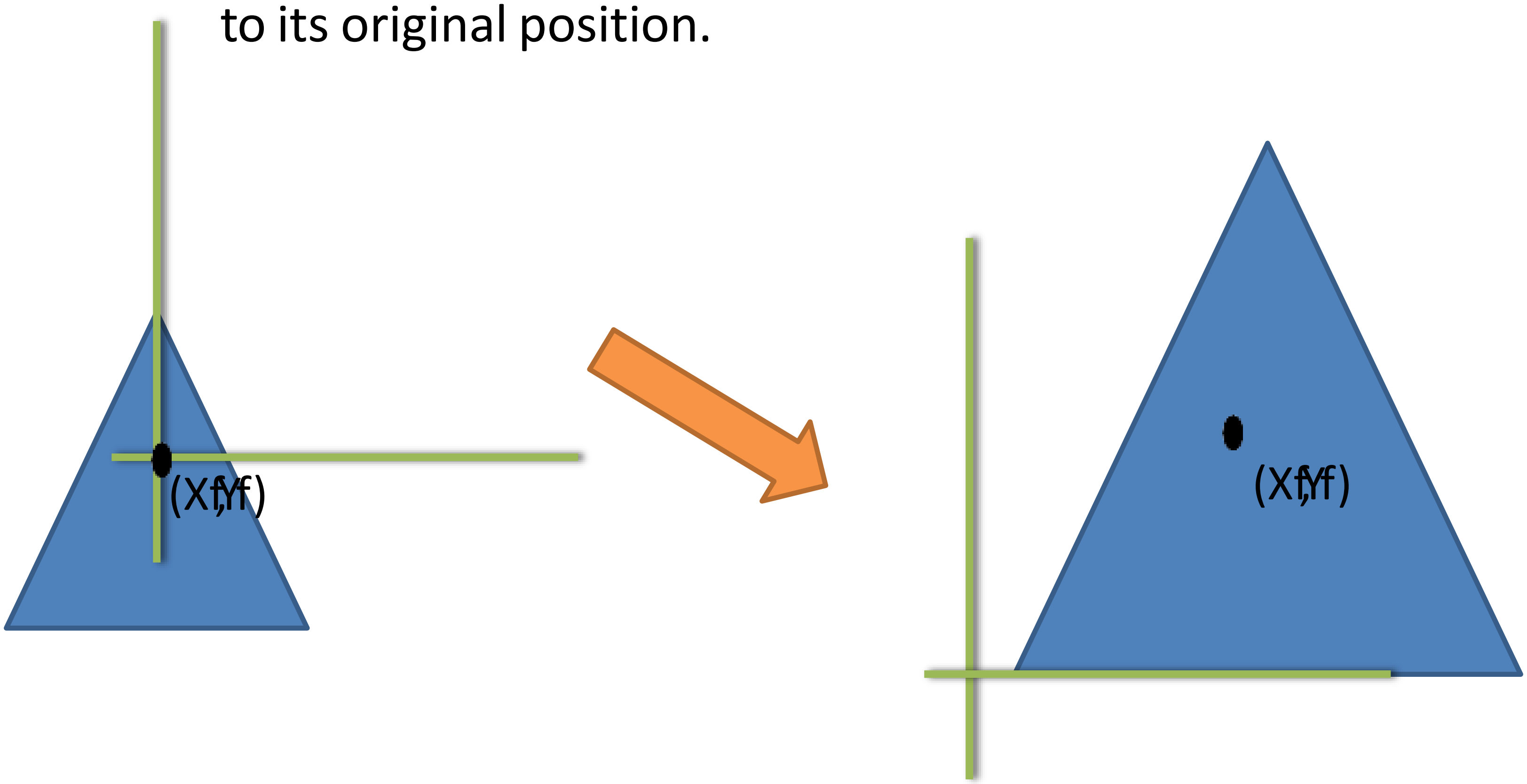
Fixed Point Scaling

Step 2 : Scaling the objet about origin



Fixed Point Scaling

Step 3: The fixed point along with the object is translated back to its original position.



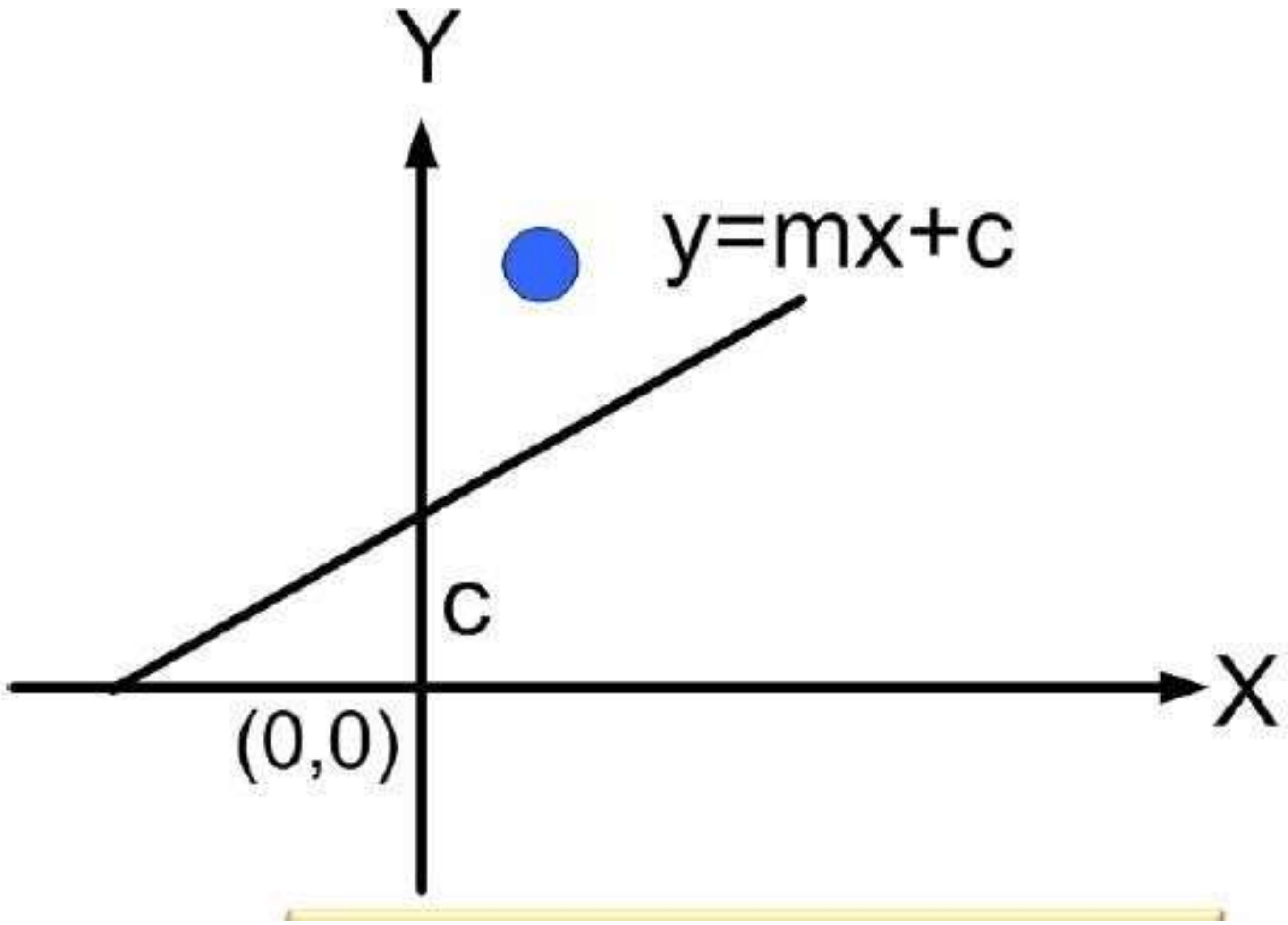
Fixed Point Scaling

Composite Transformation

$$= T'_{(Xf\ ,Yf)} \cdot S_{x,y} \quad \cdot T_{(-Xf\ ,-Yf)}$$

$$= \begin{bmatrix} 1 & 0 & Xf \\ 0 & 1 & Yf \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -Xf \\ 0 & 1 & -Yf \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about Line y=mx+c

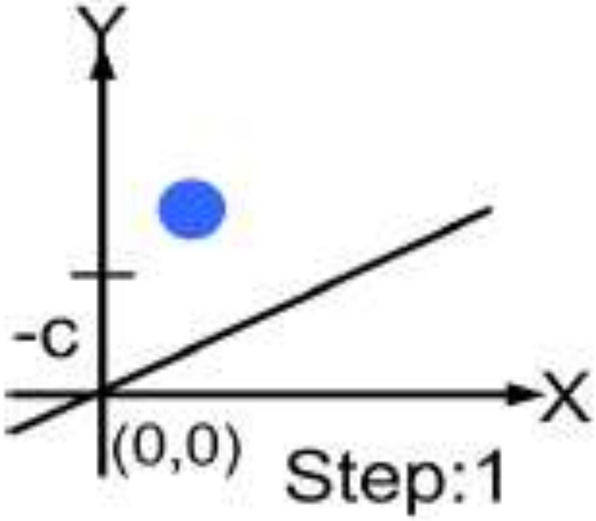


Reflection about Line y=mx+c

1. First translate the line so that it passes through the origin

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

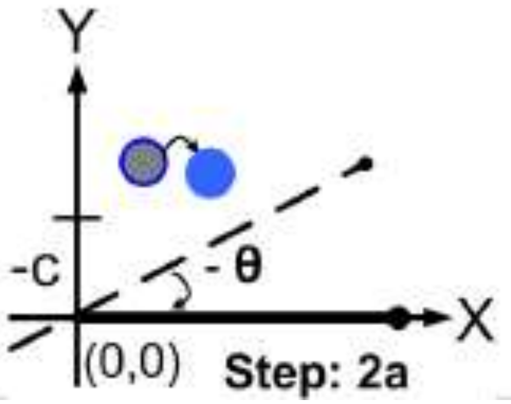
translation



Reflection about Line y=mx+c

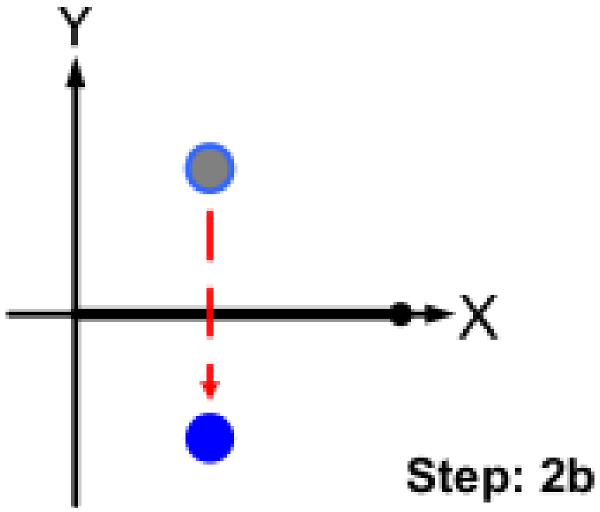
2. Rotate the line onto one of the coordinate axes(say x-axis) and reflect about that axis (x-axis)

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} --- \textit{rotation}$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reflection

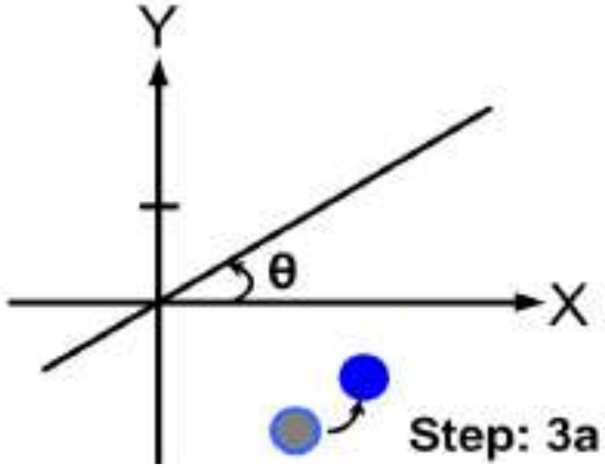


Reflection about Line y=mx+c

3. Finally, restore the line to its original position with the inverse rotation and translation transformation.

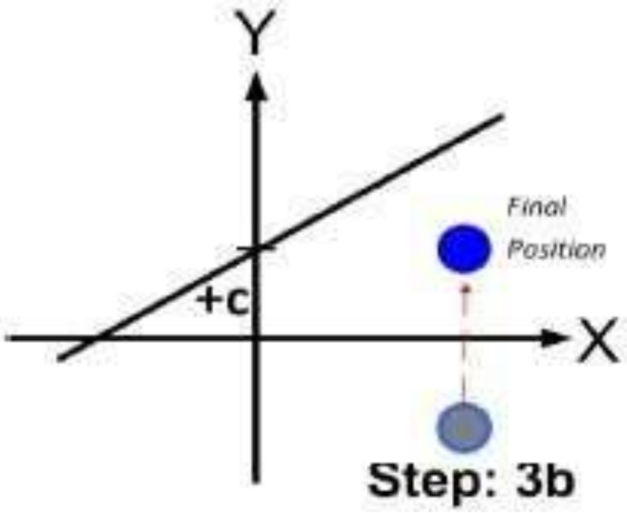
$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

translation



Reflection about Line y=mx+c

$$CM = T'_{(0,C)} \cdot R'_{\theta} \cdot R_{refl} \cdot R_{\theta} \cdot T_{(0,-C)}$$

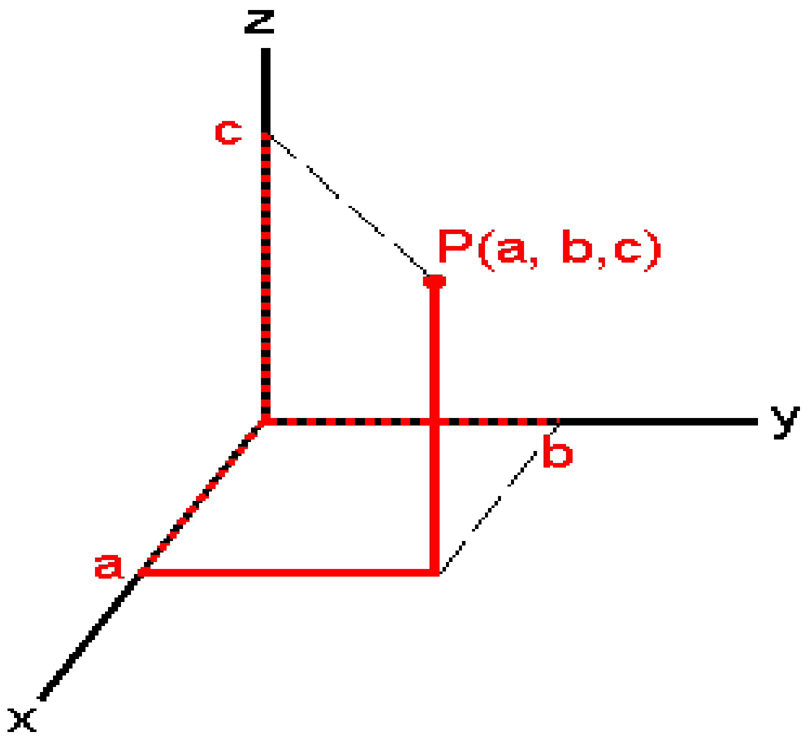
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} & \dfrac{-2cm}{1+m^2} \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{1+m^2} & \dfrac{2c}{1+m^2} \\ 0 & 0 & 1 \end{bmatrix}$$

3D Geometric Transformation

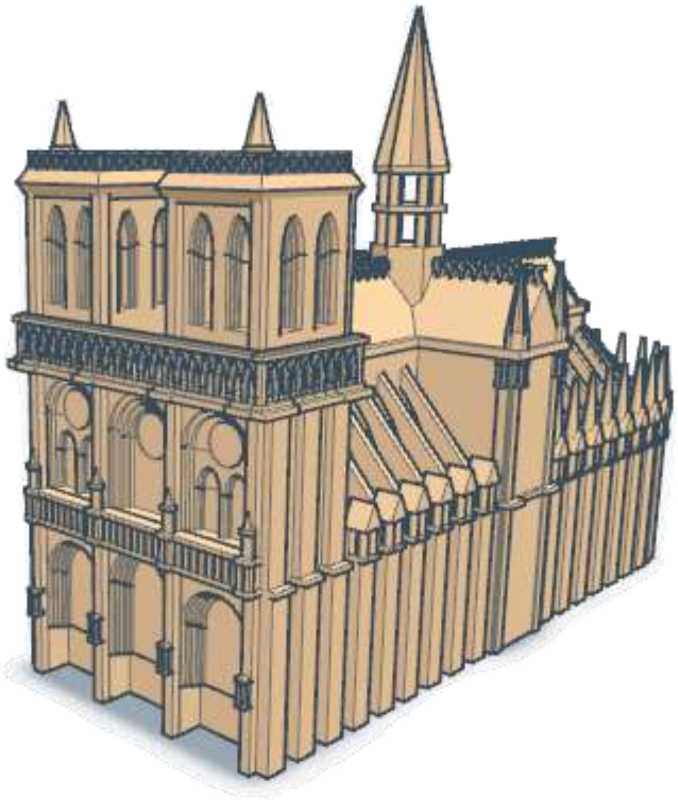
What is 3-Dimension?

- Three-dimensional space is a geometric 3-parameters model of the physical universe (without considering time) in which all known matter exists.
- These three dimensions can be labeled by a combination of length, breadth, and depth. Any three directions can be chosen, provided that they do not all lie in the same plane.



What is 3 Dimensional Object?

- An object that has length, width and depth, like any object in the real world is a 3 dimensional object.
- Types of objects: Geometrical shapes, trees, terrains, clouds, rocks, glass, hair, furniture, human body, etc.



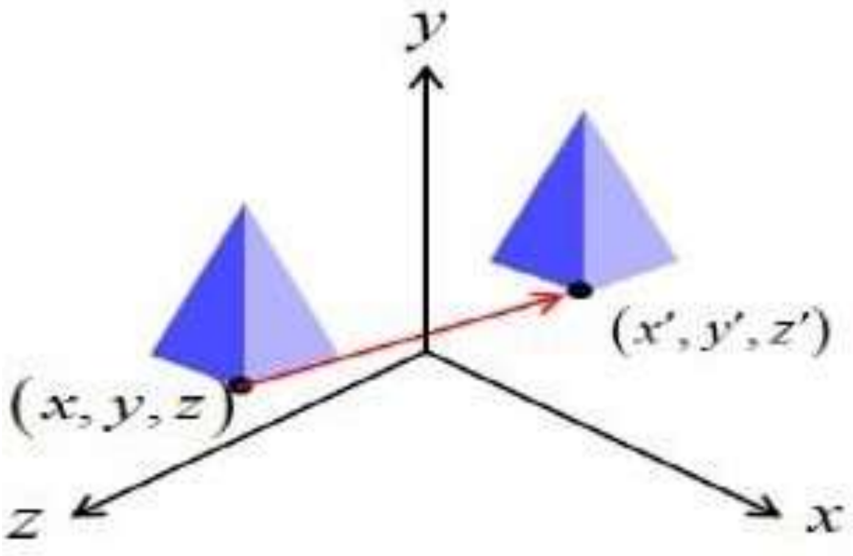
3D Transformations

- Just as 2D-transfromtion can be represented by 3x3 matrices using homogeneous co-ordinate can be represented by 4x4 matrices, provided we use homogenous co-ordinate representation of points in 3D space as well.
  1. Translation
  2. Rotation
  3. Scaling
  4. Reflection
  5. Shear

1.Translation

- Translation in 3D is similar to translation in the 2D except that there is one more direction parallel to the z-axis. If, tx, ty, and tz are used to represent the translation vectors. Then the translation of the position P(x, y, z) into the point P' (x', y', z') is done by
  - x' = x + tx
  - y' = y + ty
  - z' = z + tz

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

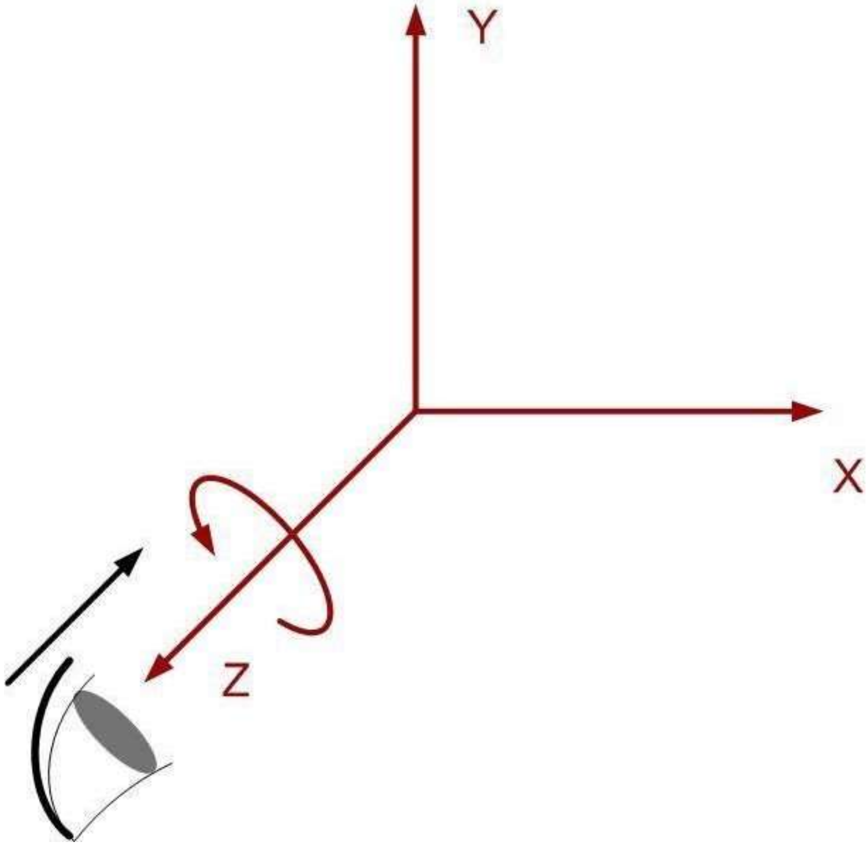


$$P' = T.P$$

- In matrix notation using homogeneous coordinate this is performed by the matrix multiplication,

2.Rotation

i) <u>Rotation About z-axis:</u>



Z-component does not change.

X' = X cosθ – Y sinθ
Y' = X sinθ + Y cosθ
Z' = Z

Matrix representation for rotation around z-axis,

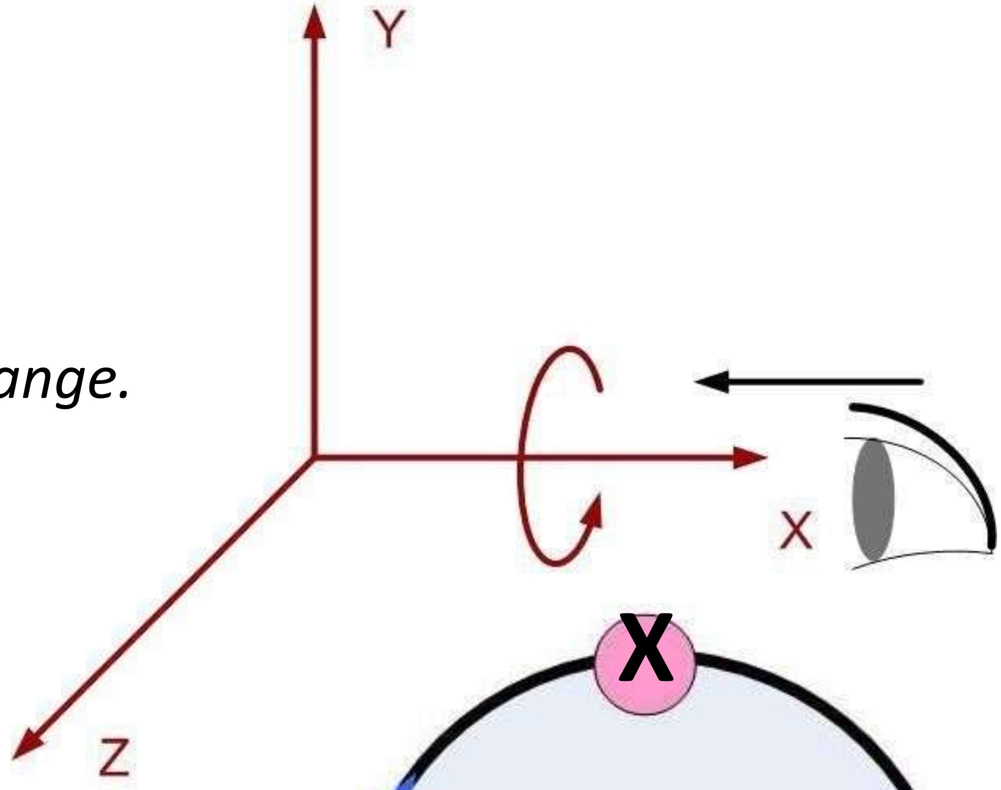
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

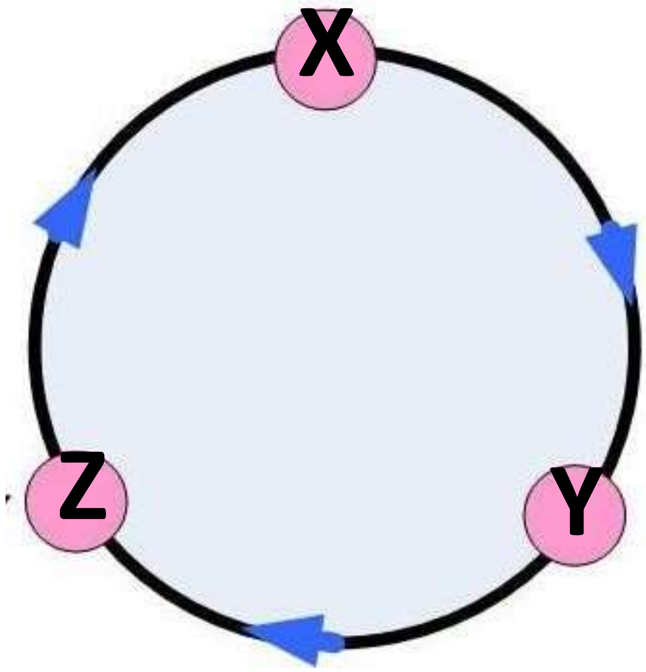
2.Rotation

ii) **Rotation About x-axis:**

X-component does not change.

Y' = Y cosθ - Z sinθ
Z' = Y sinθ + Z cosθ
X' = X



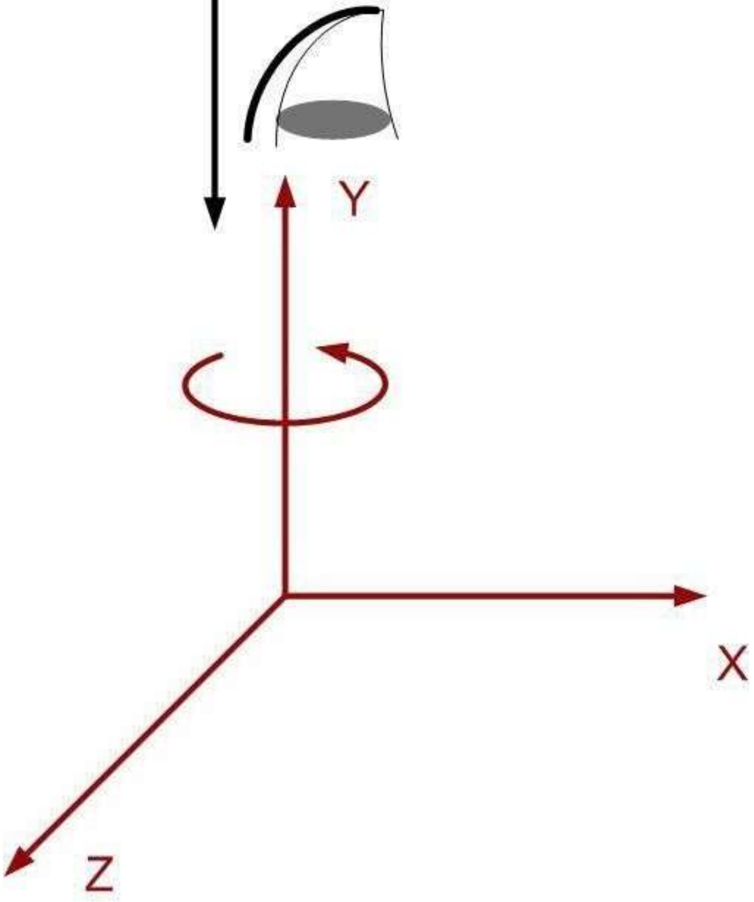


Matrix representation for rotation around X-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

2.Rotation

iii) **Rotation About Y-axis:**



Y-component does not change.

Z' = Z cosθ – X sinθ

X' = Z sinθ + X cosθ

Y' = Y

Matrix representation for rotation around Y-axis,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

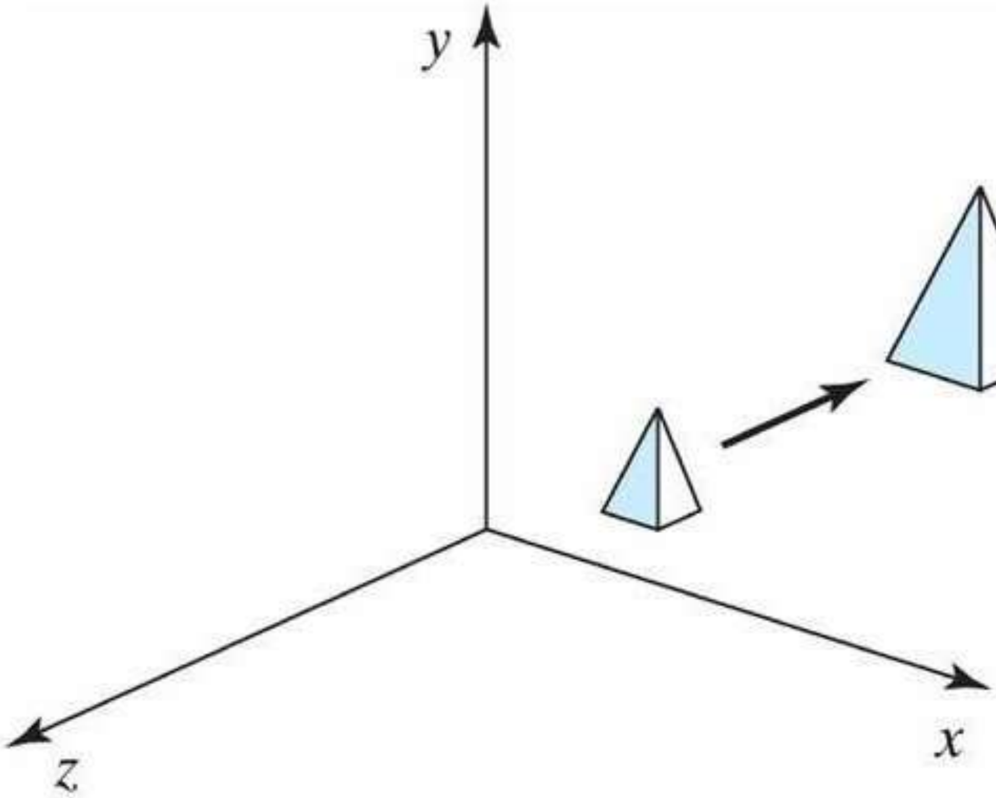
3. Scaling

Scaling about origin

X’ = X . Sx

Y’ = Y . Sy

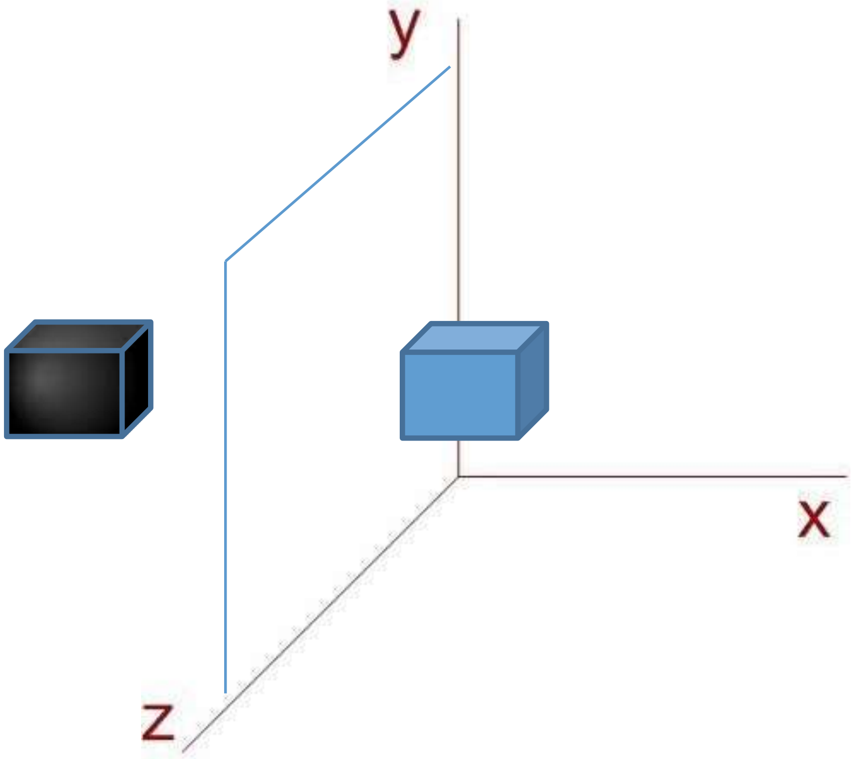
Z’ = Z . Sz



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{S} \cdot \mathbf{P}$$

4.Reflection



i) Reflection about yz plane

X’ = -X
Y’ = Y
Z’ = Z

$$T_{yz} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

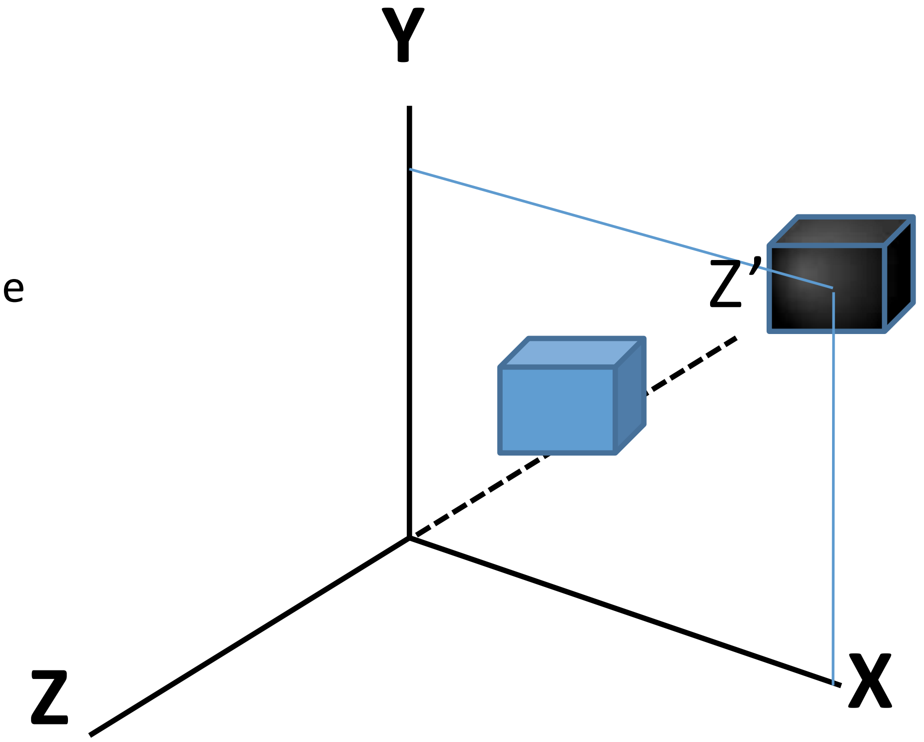
4.Reflection

ii) Reflection about XY plane

X' = X

Y' = Y

Z' = -Z



$$T_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

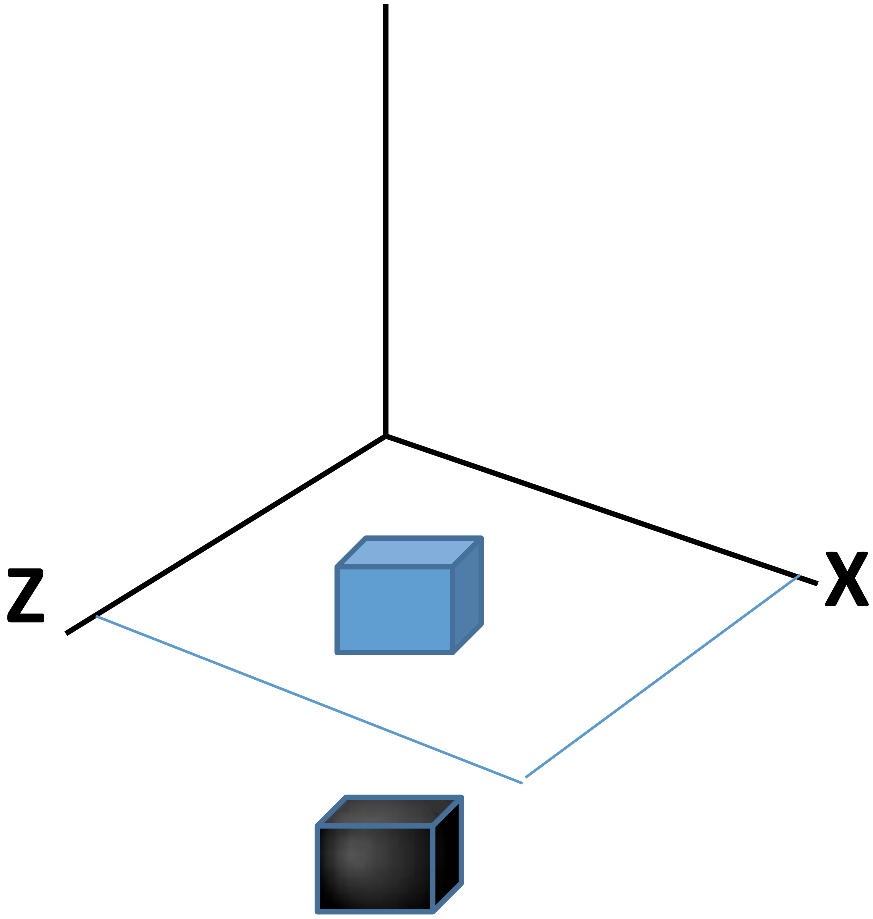
4.Reflection

iii) Reflection about XZ plane

X' = X

Y' = -Y

Z' = Z



$$T_{xz} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.Shear

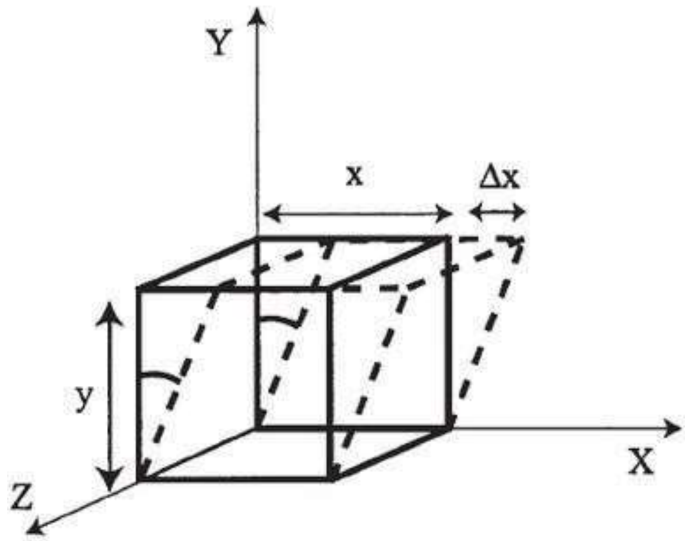
Shearing transformations can be used to modify object shapes.

Z-axis Shear

- This transformation alters x- and y-coordinate values by an amount that is proportional to the z value, while leaving the z coordinate unchanged, i.e,

$$x' = x + S_{hx} . z$$
$$y' = y + S_{hy} . z$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Similarlywe can find X-axis shear and Y-axis shear

$$SH_z = \begin{bmatrix} 1 & 0 & S_{hx} & 0 \\ 0 & 1 & S_{hy} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{hx} & 1 & 0 & 0 \\ S_{hx} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad SH_y = \begin{bmatrix} 1 & S_{hx} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & S_{hx} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$